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Fixed points of the hierarchical Potts model

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Abstract. The renormalisation group equation for the q-state Potts model having the hierarchical interaction is investigated. A stable fixed point is found to exist when $q<3$ to the leading order in $\Delta\sigma=\sigma-\frac{1}{3}$, σ being the interaction range parameter. Independent evidence suggests that there is no fixed point when $q=3$ for all $\sigma>\frac{1}{3}$. Critical exponents associated with the new fixed point are also obtained to the first order in $\Delta\sigma$.

1. Introduction

The q-state Potts model (Potts 1952) can describe various phase transitions (Fortuin and Kasteleyn 1972, Alexander and Yuval 1974, Mukamel *et al* 1976, Aharony *et al* 1977, Domany *et al* 1978) but the nature of its phase transition is not fully understood. When the spins are represented as $(q-1)$-dimensional vectors $\boldsymbol{S}_i$ (Domb 1974, Zia and Wallace 1975), the free energy as a function of the $(q-1)$-dimensional order parameter $\langle \boldsymbol{S}_i\rangle$ has the symmetry of a hypertetrahedron. It has one cubic invariant which causes the transition to be discontinuous for all $q>2$ in the mean-field approximation. However, Baxter (1973, Baxter *et al* 1978) has shown in two dimensions that the transition does not accompany latent heat for $q\leq 4$. This poses the question as to how the order of the transition changes with the dimensionality d and the number of states q. For the three-state Potts model in three dimensions, results obtained from the series expansion methods (Straley 1974, Kim and Joseph 1975, Enting and Domb 1975, Yamashita 1979) are conflicting and are inconclusive on whether the transition is continuous or not.

For the continuous-spin version of the Potts model, the Gaussian fixed point of the renormalisation group becomes unstable below six dimensions due to the presence of a cubic term in the Lagrangian. Priest and Lubensky (1976a, b) and Amit (1976) located a new fixed point which is stable in dimensions below 6 and obtained an $\epsilon=6-d$ expansion to second order for general q. The resulting value of the cubic (trilinear) coupling constant at the fixed point is proportional to $[-\epsilon/(q-q_c)]^{1/2}$ with $q_c=\frac{10}{3}$. Thus it becomes imaginary for $q>q_c$ suggesting the absence of a fixed point. This has been interpreted as the absence of a continuous transition. Even when there is a stable fixed point, the relevance of this to the phase transition of the Potts model is not clear.

In the present work, we consider the Potts model which has Dyson's hierarchical interaction (Dyson 1969, Baker 1972) instead of the usual nearest-neighbour interaction. The hierarchical interaction simulates the power-law potential decaying as $r^{-(1+\sigma)}$ and has the advantage that the partition functions possess a simple recursion relation. The interaction range parameter σ plays a role similar to that of the

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dimensionality. For the hierarchical n-vector model which has $O(n)$ symmetry (Kim and Thompson 1978), the critical value of σ which separates Gaussian from non-Gaussian behaviour is $\frac{1}{2}$. For the hierarchical Potts model, however, it is $\frac{1}{3}$. Thus one obtains a $\Delta\sigma = \sigma - \frac{1}{3}$ expansion which is analogous to the $(6-d)$ expansion.

In § 2, we define the system and transform the renormalisation group equation of the hierarchical model into an equivalent but more convenient form. We then construct, in § 3, eigenfunctions of the linearised renormalisation group operator and use these to obtain the new fixed point and its critical exponents to first order in $\Delta\sigma$. Finally, we discuss our results in § 4.

2. Renormalisation group transformations

We represent spin states of the q-state Potts model by a set of vectors which have $(q-1)$ components and write the Hamiltonian in the form

$$\mathscr{H}_L = -\sum_{i,j=1}^{N} J_{ij}\boldsymbol{S}_i \cdot \boldsymbol{S}_j - \boldsymbol{H} \cdot \sum_{i=1}^{N} \boldsymbol{S}_i \tag{1}$$

where $N = 2^L$ is the number of spins. Here the $\boldsymbol{S}_i$ have unit length and are allowed to point to q vertices of a $(q-1)$-dimensional hypertetrahedron so that $\boldsymbol{S}_i \cdot \boldsymbol{S}_j$ is unity when they are parallel and $-(q-1)^{-1}$ otherwise. The $(q-1)$-dimensional vector $\boldsymbol{H}$ is a symmetry breaking field. For the hierarchical Potts model, J_{ij} is chosen in such a way that $\mathscr{H}_L$ has the hierarchical structure;

$$\mathscr{H}_L(\boldsymbol{S}_1, \ldots, \boldsymbol{S}_N) = \mathscr{H}_{L-1}(\boldsymbol{S}_1, \ldots, \boldsymbol{S}_{N/2}) + \mathscr{H}_{L-1}(\boldsymbol{S}_{N/2+1}, \ldots, \boldsymbol{S}_N) - \lambda^{-2L}\left(\sum_{i=1}^{N} \boldsymbol{S}_i\right)^2 \tag{2}$$

with $\lambda = 2^{(1+\sigma)/2}$, σ being the interaction range parameter. Then the recursive nature of the interaction allows us to write down the exact renormalisation group transformation (RGT) as (Kim and Thompson 1978)

$$\tilde{P}_{l+1}(\lambda \boldsymbol{y}) = \pi^{-n/2} \int_{-\infty}^{\infty} \ldots \int \exp[-(\boldsymbol{x}-\boldsymbol{y})^2]\{\tilde{P}_l(\boldsymbol{x})\}^2 \, \mathrm{d}\boldsymbol{x}. \tag{3}$$

The starting point of the iteration is

$$\tilde{P}_0(\boldsymbol{x}) = \sum_{\boldsymbol{S}_i} \exp(C\boldsymbol{x} \cdot \boldsymbol{S}_i) \tag{4}$$

where $C = 2\sqrt{\beta}/\lambda$, β being the inverse temperature. The vectors appearing in equations (3) and (4) all have $n = q-1$ components. From now on, as well as in equation (3), n stands for $q-1$.

We let M_{ij} $(i = 1, 2, \ldots, q, j = 1, 2, \ldots, n)$ be the jth coordinate of ith vertex of the hypertetrahedron. Some properties of M_{ij} are (Zia and Wallace 1975)

$$\sum_{i=1}^{q} M_{ij} = 0 \tag{5}$$

$$\sum_{i=1}^{q} M_{ij}M_{ik} = \frac{q}{n}\delta_{jk} \tag{6}$$

$$\sum_{k=1}^{n} M_{ik}M_{jk} = (q\delta_{ij} - 1)/n. \tag{7}$$

We now define

$$\phi_i(\boldsymbol{x}) = (n/q)^{1/2} \sum_j M_{ij}x_j$$

$$\psi_i(\boldsymbol{y}) = (n/q)^{1/2} \sum_j M_{ij}y_j, \qquad i = 1, \ldots, q. \tag{8}$$

ϕ_i and ψ_i are not linearly independent since

$$\sum_i \phi_i = \sum_i \psi_i = 0 \tag{9}$$

due to equation (5). Equation (4) then becomes

$$\tilde{P}_0(\boldsymbol{x}) = \sum_{i=1}^{q} \exp\left(C \sum_{j=1}^{n} M_{ij}x_j\right) = \sum_{i=1}^{q} \exp[C(q/n)^{1/2}\phi_i] \equiv P_0(\boldsymbol{\phi}(\boldsymbol{x})). \tag{10}$$

$P_0(\boldsymbol{\phi})$ as defined is symmetric with regard to arbitrary exchange of its arguments. We show below that this symmetry property is preserved by the RGT and that any $\tilde{P}_l(\boldsymbol{x})$ can be written in a symmetric form in terms of $\boldsymbol{\phi}$. Suppose $\tilde{P}_l(\boldsymbol{x})$ can be expressed as

$$\tilde{P}_l(\boldsymbol{x}) = P_l(\phi_1(\boldsymbol{x}), \ldots, \phi_q(\boldsymbol{x})) \tag{11}$$

where P_l is some function which is invariant under any permutations of ϕ_i's. From equations (6) and (8), we have

$$\sum_{i=1}^{q} (\phi_i - \psi_i)^2 = \sum_{i=1}^{n} (x_i - y_i)^2.$$

Equation (5), then, can be written

$$\tilde{P}_{l+1}(\lambda \boldsymbol{y}) = \pi^{-n/2} \int_{-\infty}^{\infty} \ldots \int \exp\left(-\sum_{i=1}^{q} (\phi_i - \psi_i)^2\right) P_l^2(\phi_1, \ldots, \phi_q)\, \mathrm{d}\boldsymbol{x}. \tag{12}$$

We now change the integration variable

$$J\, \mathrm{d}\boldsymbol{x} = \mathrm{d}\phi_2, \ldots, \mathrm{d}\phi_q$$

where J is the determinant of the $n \times n$ matrix obtained from $(n/q)^{1/2}M_{ij}$ by eliminating the first row, and can be shown to be $q^{-1/2}$. Since $\Sigma\, \phi_i = 0$, equation (12) can be written in a symmetric form

$$\begin{aligned}\tilde{P}_{l+1}(\lambda \boldsymbol{y}) &= \pi^{-n/2} q^{1/2} \int_{-\infty}^{\infty} \ldots \int \exp\left(-\sum_{i=1}^{q} (\phi_i - \psi_i)^2\right) P_l^2(\phi_1, \ldots, \phi_q) \\ &\quad \times \delta\left(\sum_{i=1}^{q} \phi_i\right) \mathrm{d}\phi_1, \ldots, \mathrm{d}\phi_q \\ &\equiv P_{l+1}(\lambda\psi_1, \ldots, \lambda\psi_q). \end{aligned} \tag{13}$$

Thus $\tilde{P}_{l+1}$ is also of the form of equation (11). Replacing the Dirac delta function by its integral representation, and using equation (9), equation (13) can also be written

$$P_{l+1}(\lambda\boldsymbol{\psi}) = (q/\pi)^{1/2}\int_{-\infty}^{\infty}\exp(-qw^2)\pi^{-q/2}\int_{-\infty}^{\infty}\dots$$
$$\times\int\exp\left(-\sum_{j=1}^{q}(\phi_j-\psi_j-\mathrm{i}w)^2\right)P_l^2(\boldsymbol{\phi})\,\mathrm{d}\boldsymbol{\phi}\,\mathrm{d}w$$
$$\equiv R(\boldsymbol{\psi},\boldsymbol{\phi})\,.\,P_l^2(\boldsymbol{\phi}). \tag{14}$$

Equation (14), together with equation (9), is equivalent to equation (3) but is more convenient since the symmetry of the Potts model is manifested explicitly. If we define

$$I_k(\boldsymbol{\phi}) = \sum_{i=1}^{q}\phi_i^k, \tag{15}$$

$P_l(\boldsymbol{\phi})$ contains all possible combinations of products of I_k's except I_1 which is identically zero. If the model were of O(n) symmetry, $P_l(\boldsymbol{\phi})$ would have been a function of I_2 only. For the Potts model, we are working in a much larger parameter space. In particular the presence of I_3 in the Potts model renders it a special symmetry property.

3. Fixed point

To discuss the non-Gaussian fixed point of equation (14), we first take out the Gaussian part from P_l. The Gaussian fixed point is

$$P_{\mathrm{G}}^*(\boldsymbol{\phi}) = 2^{-\sigma n/2}\exp[\tfrac{1}{2}(1-2^{-\sigma})\phi^2] \tag{16}$$

as can be verified by direct substitution. We thus define $h_l(\boldsymbol{\phi})$ by

$$P_l(\boldsymbol{\phi}) = P_{\mathrm{G}}^*(\boldsymbol{\phi})h_l(2^{-\sigma/2}\boldsymbol{\phi}) \tag{17}$$

to obtain RGT in terms of h_l as

$$h_{l+1}(2\boldsymbol{\psi}/\lambda) = R(\boldsymbol{\psi},\boldsymbol{\phi})\,.\,h_l^2(\boldsymbol{\phi}) \tag{18}$$

where the linear operator R is defined in equation (14). Since $h_l(\boldsymbol{\phi})$ has an expansion of the form

$$h_l(\boldsymbol{\phi}) = \sum A_{k_1k_2\dots}I_{k_1}I_{k_2\dots}$$

we need to calculate

$$R(\boldsymbol{\psi},\boldsymbol{\phi})\,.\,I_{k_1}(\boldsymbol{\phi})I_{k_2}(\boldsymbol{\phi})\dots.$$

For example,

$$R(\boldsymbol{\psi},\boldsymbol{\phi})\,.\,I_2(\boldsymbol{\phi})I_2(\boldsymbol{\phi}) = I_2(\boldsymbol{\psi})I_2(\boldsymbol{\psi})+(q+1)I_2(\boldsymbol{\psi})+n(q+1)/4.$$

We write this in a short notation as

$$I_{2,2}\to I_{2,2}+(q+1)I_2+n(q+1)/4.$$

In general, we have

$$I_\alpha\to\sum_\beta d_{\alpha,\beta}I_\beta \tag{19}$$

where α, β denote a collection of indices. If α represents a collection of indices

$\{k_1, k_2, \ldots\}$, we let $|\alpha| = \Sigma_i k_i$. With this notation, $d_{\alpha,\beta} = 0$ for $|\beta| > |\alpha|$. Explicit results of the $d_{\alpha,\beta}$'s necessary for our purpose are tabulated in appendix A. From this, we can construct eigenfunctions of the linear operator $R(\frac{1}{2}\lambda\boldsymbol{\psi}, \boldsymbol{\phi})$ (see equation (18)). For a model with O(n) symmetry, they would be the generalised Laguerre polynomials (Kim and Thompson 1978). We find that the prescription for constructing eigenfunctions is

$$D_\alpha = \sum_\beta (-a^2)^{|\beta|-|\alpha|} d_{\alpha,\beta} I_\beta \tag{20}$$

where

$$a^2 = 1 - (\lambda/2)^2 = 1 - 2^{-(1-\sigma)} \tag{21}$$

and $d_{\alpha,\beta}$ is given by equation (19). We use a convention that $D_0 = 1$. D_α then satisfies

$$R(\tfrac{1}{2}\lambda\boldsymbol{\psi}, \boldsymbol{\phi}) \cdot D_\alpha(\boldsymbol{\phi}) = (\tfrac{1}{2}\lambda)^{|\alpha|} D_\alpha(\boldsymbol{\psi}). \tag{22}$$

The inverse relation of equation (20) is

$$I_\alpha = \sum_\beta (a^2)^{|\alpha|-|\beta|} d_{\alpha,\beta} D_\beta. \tag{23}$$

We do not have a general proof for equations (20), (22) and (23) but fortunately they are correct for all α which we need to consider for our calculation.

This consideration leads us to expand $h_l(\boldsymbol{\phi})$ in terms of $D_\alpha(\boldsymbol{\phi})$. Let

$$h_l^2(\boldsymbol{\phi}) = A_0\Big(1 + \sum_\alpha A_\alpha (2/\lambda)^{|\alpha|} D_\alpha(\boldsymbol{\phi})\Big) \tag{24}$$

where we have separated the constant term explicitly and the sum over α runs from D_2. We then have, from equations (18) and (22),

$$\begin{aligned} h_{l+1}^2(\boldsymbol{\psi}) &= A_0'\Big(1 + \sum_\alpha A_\alpha'(2/\lambda)^{|\alpha|} D_\alpha(\boldsymbol{\psi})\Big) \\ &= A_0^2\Big(1 + 2\sum_\alpha A_\alpha D_\alpha(\boldsymbol{\psi}) + \sum_{\alpha,\beta} A_\alpha A_\beta D_\alpha(\boldsymbol{\psi}) D_\beta(\boldsymbol{\psi})\Big). \end{aligned}$$

Next, we need to decompose $D_\alpha D_\beta$ into a linear combination of D_α's. Let

$$D_\alpha D_\beta = \sum_\gamma (\alpha; \beta \to \gamma) D_\gamma. \tag{25}$$

The coefficients $(\alpha; \beta \to \gamma)$ can be obtained using equations (20) and (23). For example,

$$D_3 D_3 = D_{3,3} - \frac{9}{2qa^2} D_{2,2} + \frac{9}{2a^2} D_4 + \frac{9(q-2)}{2qa^4} D_2 + \frac{3(q-1)(q-2)}{4qa^6}.$$

In appendix B, we tabulate these coefficients which are needed in this work. We find that $(\alpha; \beta \to \gamma)$ is non-vanishing for $||\alpha| - |\beta|| \leq |\gamma| \leq |\alpha| + |\beta|$. We then finally arrive at an algebraic RGT:

$$A_0' = A_0^2\Big(1 + \sum_{\alpha,\beta} A_\alpha A_\beta (\alpha; \beta \to 0)\Big) \tag{26}$$

$$A_0'(2/\lambda)^{|\alpha|} A_\alpha' = A_0^2\Big(2A_\alpha + \sum_{\beta,\gamma} A_\beta A_\gamma (\beta; \gamma \to \alpha)\Big). \tag{27}$$

The linearised RGT around an arbitrary fixed point A^*_α is simply given by

$$A'_\alpha - A^*_\alpha = \sum_\beta V_{\alpha,\beta}(A_\beta - A^*_\beta) \tag{28}$$

where

$$V_{\alpha,\beta} = 2A^*_0(\lambda/2)^{|\alpha|}\Big(\delta_{\alpha\beta} + \sum_\gamma A^*_\gamma(\gamma;\beta\to\alpha)\Big) - 2A^*_0 A^*_\alpha \sum_\gamma A^*_\gamma(\gamma;\beta\to 0). \tag{29}$$

For the Gaussian fixed point for which $A^*_\alpha = 0$ ($\alpha \geqslant 2$) and $A^*_0 = 1$, V is diagonal and has eigenvalues $2^{1+|\alpha|(\sigma-1)/2}$, $|\alpha| = 2, 3, \ldots$ Therefore, the Gaussian fixed point is stable for $0<\sigma<\frac{1}{3}$, and D_3 becomes marginal at $\sigma = \frac{1}{3}$. For σ close to $\frac{1}{3}$, we make an ansatz that A^*_3 is small. Then, from equation (27), A^*_2, A^*_4, $A^*_{2,2}$ and $A^*_{3,3}$ are of order $(A^*_3)^2$ while $(\sigma-\frac{1}{3})A^*_3$ is of order $(A^*_3)^3$. We therefore have a non-Gaussian fixed point for which $(A^*_3)^2$ is of order $\Delta\sigma = \sigma - \frac{1}{3}$. Explicitly we find, to the leading order in $\Delta\sigma$,

$$(A^*_3)^2 = 3^{-1}(1+2^{1/3})^{-3}\ln(2) a^6 q(3-q)^{-1}\,\Delta\sigma, \tag{30}$$

$$A^*_0 = 1 - (3;3\to 0)(A^*_3)^2, \tag{31}$$

and

$$A^*_\alpha = (2^{|\alpha|/3} - 2)^{-1}(3;3\to\alpha)(A^*_3)^2 \tag{32}$$

for $\alpha = \{2\}$, $\{4\}$, $\{2,2\}$ and $\{3,3\}$. We see that A^*_3 becomes imaginary for $q>3$. Substituting these values into equation (29), we find that the first two eigenvalues of V for this fixed point are

$$2^{1/3} + 2^{-2/3}(3-q)^{-1} q \ln(2)\,\Delta\sigma + \mathrm{O}(\Delta\sigma)^2$$

and $1 - 3\ln(2)\,\Delta\sigma + \mathrm{O}(\Delta\sigma)^2$, respectively. Therefore, this new fixed point is stable for $\sigma > \frac{1}{3}$ and is associated with critical exponents

$$1/\nu = \tfrac{1}{3} + q\Delta\sigma/2(3-q) \tag{33}$$

and

$$\gamma = \sigma\nu$$
$$= 1 + 9(2-q)\Delta\sigma/2(3-q) \tag{34}$$

to first order in $\Delta\sigma$.

We also considered the d-dimensional version of the hierarchical model where 2^d spins are grouped together to form a block spin and $\lambda = 2^{(d+\sigma)/2}$ (Blekher and Sinai 1974). The result for γ in this generalisation is identical to equation (34) with $\Delta\sigma = (\sigma/d - \frac{1}{3})$.

4. Discussion

The hierarchical Potts model is found to have a new fixed point stable for $\sigma \geqslant \frac{1}{3}$. As in the short-ranged continuous spin version of the Potts model, this new fixed point exists for $q < q_c$ where $q_c = 3$ to the leading order. Recently, Guim and Kim (1980) studied equation (3) numerically for $q = 3$ and found that there is no temperature at which $\tilde{P}_l(\boldsymbol{x})$ approaches a fixed point as $l \to \infty$ for $\frac{1}{3} < \sigma < 1$. This failure to observe a fixed point at

$q=3$ is consistent with our present result. However, they also found that the high-temperature susceptibility diverges with the usual power-law behaviour at the critical temperature even though there is no fixed point. In view of this, it is difficult to relate the absence of a stable fixed point for $q \geq 3$ and for $\sigma > \frac{1}{3}$ to the absence of a continuous phase transition.

In the range $0 < \sigma < \frac{1}{3}$, where the Gaussian fixed point is stable, Guim and Kim found numerically that $P_l(\boldsymbol{\phi}) \to P_G^*$ at the critical temperature and that the critical behaviours are classical. Thus, the Gaussian fixed point is globally stable in the Potts model. It has been generally believed that the predictions of mean-field theory are correct for the system of high enough dimensionalities or for those having a long enough range of interaction (Gates and Thompson 1975). But this expectation does not hold here.

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Appendix A. The coefficients $d_{\alpha,\beta}$ used in this work (see equation (19))

$$I_k \to \sum_{m=0}^{k/2} (n/q)^m [k!/(k-2m)!m!4^m] I_{k-2m} \qquad (I_1 \equiv 0, I_0 = q, n = q-1)$$

$$I_{2,2} \to I_{2,2} + (q+1)I_2 + n(q+1)/4$$

$$I_{2,3} \to I_{2,3} + (q+5)/2I_3$$

$$I_{2,4} \to I_{2,4} + 3n/qI_{2,2} + (q+7)/2I_4 + 9n(q+3)/4qI_2 + 3n^2(q+3)/8q$$

$$I_{3,3} \to I_{3,3} + 9/2I_4 - 9/2qI_{2,2} + 9(q-2)/2qI_2 + 3n(q-2)/4q$$

$$I_{3,4} \to I_{3,4} + 3(q-3)/qI_{2,3} + 6I_5 + 3(q^2+22q-35)/4qI_3$$

$$I_{2,2,2} \to I_{2,2,2} + 3(q+3)/2I_{2,2} + 3(q+3)(q+1)/4I_2 + (q+3)(q+1)n/8$$

$$I_{2,2,3} \to I_{2,2,3} + (q+7)I_{2,3} + (q+5)(q+7)/4I_3$$

$$I_{2,3,3} \to I_{2,3,3} + (q+11)/2I_{3,3} + 9/2I_{2,4} - 9/2qI_{2,2,2} + 9(q+9)/4I_4 + 9(q-13)/4qI_{2,2} + 3(q^2+3q-10)/qI_2 + 3n(q^2+3q-10)/8q$$

$$I_{3,3,3} \to I_{3,3,3} - 27/2qI_{2,2,3} + 27/2I_{3,4} + 27(q-8)/2qI_{2,3} + 81/2I_5 + 9(q^2+27q-70)/4qI_3.$$

Appendix B. The coefficients $(\alpha; \beta \to \gamma)$ used in this work (see equation (25))

$(2; 2 \to 2) = 2/a^2$
$(2; 3 \to 3) = 3/a^2$
$(2; 3 \to 2, 3) = 1$
$(2; 4 \to 2) = 3(q-1)/qa^4$

$(3; 3 \to 4) = 9/2a^2$
$(3; 3 \to 2, 2) = -9/2qa^2$
$(3; 3 \to 3, 3) = 1$
$(3; 4 \to 3) = 9(q-2)/qa^4$

$(2;2,2\to 2)=(q+1)/a^4$
$(2;3,3\to 2)=0$
$(3;3\to 0)=3(q-1)(q-2)/4qa^6$
$(3;3\to 2)=9(q-2)/2qa^4$
$(3;2,2\to 3)=6/a^4$
$(3;2,3\to 2)=3(q-2)(q+5)/4qa^6$
$(3;3,3\to 3)=3(q-2)(q+8)/2qa^6$

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